Global bifurcation of traveling waves in discrete nonlinear Schrödinger equations

Carlos García-Azpeitia*

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Abstract

The discrete nonlinear Schrödinger equations of n sites are studied with periodic boundary conditions. These equations have n branches of standing waves that bifurcate from zero. Traveling waves appear as a symmetry-breaking from the standing waves for different amplitudes. The bifurcation is proved using the global Rabinowitz alternative in subspaces of symmetric functions. Applications to the Schrödinger and Saturable lattices are presented.

Keywords: NLS-like equations; Periodic solutions; Symmetries, Equivariant bifurcation; Degree theory

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1 Introduction

The discrete nonlinear Schrödinger equation (DNLS) appears in the study of optical waveguide arrays and Bose–Einstein condensates trapped in optical lattices [10]. In this paper, we consider a general lattice of n sites described by the equations

$$i\dot{q}_j = V'(|q_j|^2)q_j + (q_{j+1} - q_j) + (q_{j-1} - q_j), \tag{1}$$

*Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, 04510 Ciudad de México, México. E-mail: cgazpe@ciencias.unam.mx

where the sites, $q_j(t) \in \mathbb{C}$ for $j = 1, ..., n$, satisfy the periodic boundary conditions $q_j = q_{j+n}$. The equations include the Schrödinger lattice, $V(x) = cx^2/2$, and the Saturable lattice, $V(x) = c\ln(1+x)$, where c represents the strength of the linear coupling after rescaling.

Equations (1) have the explicit solutions

$$q_j(t) = ae^{i\omega t}e^{ijm\zeta} \text{ with } \zeta = \frac{2\pi}{n}, \tag{2}$$

for $m = 1, ..., n$, where the frequency ω is a function of the amplitude $a \in \mathbb{R}^+$ given in (6). These solutions are relative equilibria and are known as standing waves in the sense that their norms are stationary in time.

We prove existence of periodic solutions using a non-abelian group that acts by permutating the oscillators and shifting and reflecting phase and time (see Definition 1). In fact, solutions (2) appear from symmetry-breaking of the trivial solution; from them, we prove a secondary symmetry-breaking of periodic solutions (Theorem 5).

Main result. Let $m \in [0, n/2] \cap \mathbb{N}$ with $m \neq n/4$, for each $k \in [1, n/2] \cap \mathbb{N}$ such that

$$\phi_k(a) \in (-\infty, 1) \backslash \{\gamma_j : j \in [1, n/2] \cap \mathbb{N}\}, \tag{3}$$

the relative equilibrium (2) has two global bifurcations of solutions of the form

$$q_j(t) = e^{i\omega t}e^{ijm\zeta}\left(a + x\left(\nu t \pm jk\zeta\right)\right), \tag{4}$$

where

$$\phi_k(a) = \frac{a^2V''(a^2)}{2\cos m\zeta \sin^2 \frac{k\zeta}{2}} \qquad \text{and} \qquad \gamma_k = 1 - \cot^2\frac{k\zeta}{2}\tan^2 m\zeta.$$

Each branch is a global continuum in the space of 2π-periodic functions x and frequencies ν emanating from $(0, \nu_k^{\pm})$ given in (17).

These solutions are traveling waves in the sense that their norms satisfy

$$\left|q_j\right|(t) = a + r(\nu t \pm jk\zeta),$$

where $r(t)$ is real 2π-periodic; these solutions are known as traveling or moving breathers when they are localized. The existence of localized traveling waves in infinite lattices is proved in [18] (see also Chapter 16 and 21 in [10] and [3]).

In Theorem 7 we prove that solutions (2) are stable if the conditions (3) hold for $k = 1, ..., n-1$. In [10], solutions given by (2) are called plane waves, the nonlinear dispersion relation (6.7) is equivalent to (6) and the modulation stability (6.8) has stable directions precisely when (3) holds.

In [7], the authors find a bifurcation of relative equilibria for $m = 1$ and amplitudes $\phi_k(a) = \gamma_k$. This bifurcation exists due to an eigenvalue of the linearization that crosses zero. This phenomena occurs for any m, and we should expect a bifurcation of standing waves appearing from the amplitudes $\phi_k(a) = \gamma_k$.

For $\phi_k(a) \geq 1$, there are two eigenvalues colliding on the imaginary axis and detaching into the complex plane. This phenomenon may trigger a Hamiltonian-Hopf bifurcation where isolated traveling waves persist for $\phi_k(a) > 1$. Indeed, a bifurcation of this kind is described in [9] for the trimer $n = 3$.

The authors prove in [4, 6, 5] bifurcation of periodic solutions for bodies, vortices and Schrödinger sites, for $m = 1$. In the body and vortex problems, the coupling is homogeneous and invariant under all permutations, i.e. the stability and bifurcation properties of (2) are independent of a and m; but this is not the case for Schrödinger sites. We complete the analysis of bifurcation and stability for all m's.

References [4, 6, 5] use a global Lyapunov-Schmidt reduction and a topological degree for G-equivariant maps that are orthogonal to the generators (see [1, 2, 8]). In this paper we present a direct and self-contained approach, requiring non-abelian group actions and the global Rabinowitz alternative [19].

In Section 2, we define the equivariant properties of the bifurcation operator, we present a reduction to a finite number of Fourier components, and we find the spectra. In Section 3.1 we analyze the spectra for each irreducible representation and we prove the bifurcation result. In Section 3.2 we study the stability. In Section 4, we apply the results to the focusing and defocusing Schrödinger and Saturable lattices.

2 Setting the problem

Equations (1) in rotating coordinates, $q_j(t) = e^{i\omega t}u_j(t)$, are

$$i\dot{u}_j - \omega u_j = V'(|u_j|^2)u_j + (u_{j+1} - u_j) + (u_{j-1} - u_j)\,. \tag{5}$$

The values $a_j = ae^{ijm\zeta}$ satisfy $a_j = a_{j+n}$ and

$$(a_{j+1} - 2a_j + a_{j-1}) = (-4\sin^2 m\zeta/2)a_j.$$

Then $u_j(t) = a_j$ is an equilibrium and (2) is a solution of (1), when

$$\omega = 4\sin^2 m\zeta/2 - V'(a^2). \tag{6}$$

In real coordinates, $u_j \in \mathbb{R}^2$, equations (5) are

$$J\dot{u}_j = \omega u_j + V'(|u_j|^2)u_j + (u_{j+1} - 2u_j + u_{j-1}),$$

where J is the symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Let $u = (u_1, ..., u_n)$ and $\mathcal{J} = diag(J, ..., J)$. The system of equations in vectorial form is

$$\mathcal{J}\dot{u} = \nabla H(u),$$

where H is the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n}\left\{V(|u_j|^2) + \omega|u_j|^2 - |u_{j+1} - u_j|^2\right\}. \tag{8}$$

In this setting, the relative equilibrium is

$$\mathbf{a}_m = (a_1, ..., a_n) \text{ with } a_j = ae^{jm\zeta J}e_1. \tag{9}$$

Using the change of variables $u(t) = \mathbf{a}_m + x(\nu t)$, $2\pi/\nu$-periodic solutions of the Hamiltonian system correspond to zeros of the operator

$$f(x;\nu) = \mathcal{J}\dot{x} - \nu^{-1}\nabla H(\mathbf{a}_m + x) : H^1_{2\pi}(\mathbb{R}^{2n}) \times \mathbb{R}^+ \to L^2_{2\pi}(\mathbb{R}^{2n}).$$

Below we will prove global bifurcation of periodic solutions from the set of trivial solutions $(0, \nu)$ for $\nu \in \mathbb{R}^+$.

Definition 1 *Let m be as in (9). The linear action of the non-abelian group*

$$G = (\mathbb{Z}_n \times S^1) \cup \kappa(\mathbb{Z}_n \times S^1)$$

in $L^2_{2\pi}(\mathbb{R}^{2n})$ is given by the homomorphism of groups $\rho : G \to GL(L^2_{2\pi})$ generated by

$$\rho(\zeta,\varphi)x_j(t) = e^{-m\zeta J}x_{j+1}(t+\varphi),$$
$$\rho(\kappa)x_j(t) = Rx_{n-j}(-t),$$

where $\zeta = 2\pi/n$ generates $\mathbb{Z}_n$ in $S^1 = [0,2\pi)$ and κ is the reflection that generates two copies of $\mathbb{Z}_n \times S^1$ in G.

Since the Hamiltonian H is gauge invariant and autonomous, the map f is G-equivariant. Moreover, all the elements of G leave the equilibrium $\mathbf{a}_m$ fixed, and then the isotropy group of $\mathbf{a}_m$ is $G_{\mathbf{a}_m} = G$.

2.1 Irreducible representations

In this section we find the irreducible representations of the action of G. We define the isomorphisms $T_k : \mathbb{C}^2 \to V_k$ as

$$T_k z = n^{-1/2}(e^{(ikI+mJ)\zeta}z, ..., e^{n(ikI+mJ)\zeta}z) \in \mathbb{C}^{2n}, \tag{10}$$

where V_k is the image of T_k for $k \in \{1,...,n\}$.

A function $x(t) \in L^2_{2\pi}(\mathbb{R}^{2n})$ is expressed in the Fourier components as $x = \sum_{l\in\mathbb{Z}} x_l e^{ilt}$, and each Fourier component $x_l \in \mathbb{C}^{2n}$ as $x_l = \sum_{k=1}^n T_k x_{k,l}$ with $x_{k,l} \in \mathbb{C}^2$. Therefore, a function can be expressed in the orthonormal coordinates $T_k x_{k,l} e^{ilt}$ as

$$x(t) = \sum_{(k,l)\in\mathbb{Z}_n\times\mathbb{Z}} T_k x_{k,l} e^{ilt}, \qquad \mathbb{Z}_n = \{1,...,n\}. \tag{11}$$

Let us denote the j-th component of $T_k x_{k,l} \in \mathbb{C}^{2n}$ by the two-dimensional vector $n^{-1/2}e^{j(ikI+mJ)\zeta}x_{k,l} \in \mathbb{C}^2$. With this notation, the j-th component of $\rho(\zeta)T_k x_{k,l}$ is

$$n^{-1/2}e^{-mJ\zeta}e^{(j+1)(ikI+mJ)\zeta}x_{k,l} = n^{-1/2}e^{j(ikI+mJ)\zeta}(e^{ik\zeta}x_{k,l}),$$

then

$$\rho(\zeta)T_k x_{k,l} = e^{ik\zeta}T_k x_{k,l}.$$

Similarly,

$$\rho(\kappa)T_k x_{k,l} = T_k R\bar{x}_{k,l}.$$

Therefore, the subspaces of similar irreducible representations of the space $L^2_{2\pi}(\mathbb{R}^{2n})$ by the action of G are

$$V_{k,l} = \{T_k x_{k,l} e^{ilt} : x_{k,l} \in \mathbb{C}^2\}.$$

In the components $x_{k,l}$, the action of G is

$$\rho(\zeta)x_{k,l} = e^{ik\zeta}x_{k,l}, \qquad \rho(\varphi)x_{k,l} = e^{il\varphi}x_{k,l}, \qquad \rho(\kappa)x_{k,l} = R\bar{x}_{k,l}. \tag{12}$$

2.2 Linearization

Since $V_{k,l}$ are the subspaces of similar irreducible representations of G and

$$f'(0) = \mathcal{J}\partial_t - \nu^{-1}D^2H(\mathbf{a}_m)$$

is G-equivariant, by Schur's lemma, the linearization $f'(0)$ is block diagonal in the components $V_{k,l}$. The diagonal decomposition can be obtained explicitly using the following proposition,

$$f'(0)x = \sum_{(k,l)\in\mathbb{Z}_n\times\mathbb{Z}} T_k(ilJ - B_k)x_{k,l}e^{ilt}.$$

Proposition 2 *Let α_k and β_k be*

$$\alpha_k = 4\cos m\zeta \sin^2 k\zeta/2, \qquad \beta_k = 2\sin m\zeta \sin k\zeta, \tag{13}$$

the Hessian $D^2H(\mathbf{a}_m)$ satisfy that

$$D^2H(\mathbf{a}_m)T_k z = T_k B_k z,$$

where B_k is the 2×2 matrix

$$B_k = \mathrm{diag}(2a^2V''(a^2) - \alpha_k, -\alpha_k) + iJ\beta_k. \tag{14}$$

Proof. We express $D^2H(\mathbf{a}_m)$ in 2×2 blocks $A_{i,j}$ as

$$D^2H(\mathbf{a}_m) = (A_{i,j})_{i,j=1}^n.$$

Since the coupling in the DNLS equations happens only between adjacent sites, then $A_{i,j} = I$ for $|i-j| = 1$ and $A_{i,j} = 0$ for $|i-j| > 1$, modulus n.

Using $a_j = ae^{jm\zeta J}e_1$, we have

$$\frac{1}{2}D^2V(\mathbf{a}_m) = \ V'(a^2)I + 2a^2V''(a^2)e^{(jm\zeta)J}e_1e_1^Te^{-(jm\zeta)J},$$

where $e_1e_1^T = diag(1,0)$. Since $V'(a^2) + \omega - 2 = -2\cos m\zeta$, we conclude

$$A_{j,j} = -2(\cos m\zeta)I + 2a^2V''(a^2)e^{(jm\zeta)J}e_1e_1^Te^{-(jm\zeta)J}.$$

Given that the j-th component of $T_kz \in \mathbb{C}^{2n}$ is $n^{-1/2}e^{j(ikI+mJ)\zeta}z \in \mathbb{C}^2$, where $z \in \mathbb{C}^2$, the j-th component of $D^2H(a)T_kz$ is

$$\frac{1}{\sqrt{n}}\left(A_{j,j} + e^{(ikI+mJ)\zeta} + e^{-(ikI+mJ)\zeta}\right)e^{j(ikI+mJ)\zeta}z = \frac{1}{\sqrt{n}}e^{j(ikI+mJ)\zeta}\left(B_kz\right),$$

where

$$B_k = -2(\cos m\zeta)I + 2a^2V''(a^2)e_1e_1^T + e^{(ikI+mJ)\zeta} + e^{-(ikI+mJ)\zeta}.$$

From the equalities

$$e^{(ikI+mJ)\zeta} + e^{-(ikI+mJ)\zeta} = (2\cos k\zeta\cos m\zeta)I + (2\sin k\zeta\sin m\zeta)iJ,$$

and

$$-2\cos m\zeta(1-\cos k\zeta) = -4\cos m\zeta(\sin k\zeta/2)^2 = -\alpha_k,$$

we conclude that

$$B_k = -\alpha_kI + \beta_k(iJ) + 2a^2V''(a^2)diag(1,0).$$

■

3 Main Results

The symmetries permit us to assume, without loss of generality, that $m \in [0,n/2]\cap\mathbb{N}$. Throughout this section we also assume $m \neq n/4$.

In these cases the sign of α_k is well defined for $k = 1,...,n-1$,

$$sgn(\alpha_k) = \begin{cases} 1 \text{ if } m \in [0,n/4) \\ -1 \text{ if } m \in (n/4,n/2] \end{cases},$$

and we can define

$$\phi_k(a) = \frac{2a^2}{\alpha_k} V''(a^2), \qquad \gamma_k = 1 - \left(\frac{\beta_k}{\alpha_k}\right)^2. \tag{15}$$

Therefore, the matrix $D^2 H(\mathbf{a}_m)$ is block diagonal with blocks

$$B_n = diag(2a^2 V''(a^2), 0), \qquad B_k = \alpha_k diag(\phi_k - 1, -1) + \beta_k (iJ),$$

for $k = 1, .., n-1$.

3.1 Bifurcation theorem

We consider bifurcation in the fixed point space of the isotropy group $\tilde{D}_n$ generated by $(\zeta, -k\zeta)$ and κ. Solutions in the fixed point space of $\tilde{D}_n$ have symmetries

$$x_j(t) = \rho(\zeta, -k\zeta) x_j(t) = e^{-m\zeta J} x_{j+1}(t - k\zeta). \tag{16}$$

That is $x_j(t) = e^{jm\zeta J} x_n(t + jk\zeta)$ and, by the action of κ,

$$x_n(t) = \rho(\kappa) x_n = R x_n(-t).$$

The component $x_{k,1} \in \mathbb{C}^2$ is fixed by $\kappa \in \tilde{D}_n$ if $x_{k,1} \in \mathbb{R} \times i\mathbb{R}$.

Lemma 3 *The matrix iJB_k in the subspace $\mathbb{R} \times i\mathbb{R}$ has eigenvalues*

$$\nu_k^{\pm} = \beta_k \pm \sqrt{\alpha_k^2 (1 - \phi_k)}. \tag{17}$$

Moreover,

(a) *If $k \in [1, n-1] \cap \mathbb{N}$ and $\phi_k(a) \in (-\infty, \gamma_k)$, then ν_k^+ is positive.*

(b) *If $k \in [1, n/2) \cap \mathbb{N}$ and $\phi_k(a) \in (\gamma_k, 1)$, then ν_k^+ and ν_k^- are positive.*

Proof. Let $L = \mathrm{diag}(1, i)$. The eigenvalues of iJB_k in the subspace $\mathbb{R} \times i\mathbb{R}$ are the eigenvalues of the real matrix

$$L^{-1}(iJB_k)L = \begin{pmatrix} \beta_k & -\alpha_k \\ \alpha_k(\phi_k - 1) & \beta_k \end{pmatrix}.$$

The eigenvalues ν of this matrix are the zeros of

$$d_k(\nu) = (\nu - \beta_k)^2 - \alpha_k^2 (1 - \phi_k). \tag{18}$$

The function $d_k(0)$ has two real solutions if and only if $\phi_k \in (-\infty, 1)$. Since $d_k(\nu)$ is a polynomial of order ν^2 at infinity, then ν_k^+ is positive and ν_k^- is negative if $d_k(0)$ is negative. This is the case if $\phi_k < 1 - (\beta_k/\alpha_k)^2 = \gamma_k$. For $\phi \in (\gamma_k, 1)$, the function $d_k(0)$ is positive, and the values $\nu_k^\pm$ have the same sign of β_k. We conclude this result from the fact that $\beta_k > 0$ for $k \in [1, n/2) \cap \mathbb{N}$. ■

Definition 4 *We say that the amplitude a is non-degenerate if $V''(a^2) \neq 0$, $\phi_k(a) \neq \gamma_k$ for $k = 1, ..., n-1$. We say that the frequency $\nu_k^\pm$ for $k = 1, ..., n-1$ is non-resonant if $\nu_j^\pm \neq l\nu_k^\pm$ for integers $l \geq 1$ and $j \neq k$.*

The matrix B_n restricted to the subspace $x_{n,0} = \rho(\kappa)x_{n,0} = Rx_{n,0}$ has the simple eigenvalue $2a^2V''(a^2)$. For $k = 1, ..., n$, the blocks B_k have determinants $\beta_k^2 - \alpha_k^2(1 - \phi_k)$. Therefore, the non-degeneracy property of a assures that $D^2H(\mathbf{a}_m)$ has no zero-eigenvalues in $\mathrm{Fix}(\tilde{D}_n)$.

Theorem 5 *Assume that a is non-degenerate. If $\nu_k^\pm$ is non-resonant, then*

(a) *For each $k \in [1, n-1] \cap \mathbb{N}$ such that $\phi_k(a) \in (-\infty, \gamma_k)$, there is a global bifurcation from $(0, \nu_k^+)$ in the space*

$$\{x \in H_{2\pi}^2 : x_j(t) = e^{jm\zeta J} x_n(t + jk\zeta), x_n(t) = Rx_n(-t)\} \times \mathbb{R}^+ \tag{19}$$

(b) *For $k \in [1, n/2] \cap \mathbb{N}$ such that $\phi_k(a) \in (\gamma_k, 1)$, there are global bifurcations from $(0, \nu_k^+)$ and $(0, \nu_k^-)$ in the space given by (19).*

Proof. Let $K : L_{2\pi}^2 \to H_{2\pi}^1$ be the operator defined in the Fourier basis $x = \sum_{l \in \mathbb{Z}} x_l e^{ilt}$ as

$$Kx = x_0 + \sum_{l \in \mathbb{Z} \setminus \{0\}} (li\mathcal{J})^{-1} x_l e^{ilt}.$$

Since $K : H_{2\pi}^1 \to H_{2\pi}^1$ is compact and

$$f(x) = \mathcal{J}\dot{x} - \nu^{-1} D^2H(\mathbf{a}_m)x + \mathcal{O}(|x|^2), \tag{20}$$

then

$$Kf(x) = x - T(\nu)x + g(x) : H_{2\pi}^1 \to H_{2\pi}^1, \tag{21}$$

where $T(\nu)$ is the compact linear operator

$$Tx = (I + \nu^{-1} D^2H(\mathbf{a}_m))x_0 + \sum_{l \in \mathbb{Z} \setminus \{0\}} (\nu l)^{-1} i\mathcal{J} D^2H(\mathbf{a}_m) x_l e^{ilt},$$

and $g(x) = \mathcal{O}(|x|^2_{H^1_{2\pi}})$ is a nonlinear compact operator.

Since f and K are G-equivariant, the operator Kf is G-equivariant. Then Kf is well defined in the space $\mathrm{Fix}(\tilde{D}_n)$ given in (19). The global bifurcation follows from Theorem 3.4.1 in [14] if $T(\nu)$ has a simple eigenvalue crossing 1 in $\mathrm{Fix}(\tilde{D}_n)$.

For $l = 0$, by the non-degeneracy property of a, the matrix $I+\nu^{-1}D^2H(\mathbf{a}_m)$ has no eigenvalues equal to 1 in $\mathrm{Fix}(\tilde{D}_n)$. For $l \geq 2$, due to the non-resonance property of $\nu_k^{\pm}$, the matrix $(l\nu_k^{\pm})^{-1}i\mathcal{J}D^2H(\mathbf{a}_m)$ has no eigenvalues equal to 1. For $l = 1$, we have that $\nu^{-1}i\mathcal{J}D^2H(\mathbf{a}_m)$ has an eigenvalue crossing 1 when ν crosses $\nu_k^{\pm}$, corresponding to the block iJB_k. Moreover, this eigenvalues is simple when iJB_k is restricted to $\mathbb{R} \times i\mathbb{R}$. We conclude that $T(\nu)$ in $\mathrm{Fix}(\tilde{D}_n)$ has a simple eigenvalue crossing 1 when ν crosses $\nu_k^{\pm}$. ■

Remark 6 *In the case of* $1 : l$ *resonances,* $l\nu_k^{\pm} = \nu_j^{\pm}$ *for* $l \geq 2$*, the previous theorem gives the existence of the bifurcation with the biggest frequency* $\nu_j^{\pm}$*. In the case of* $1 : 1$ *resonances,* $\nu_k^{+} = \nu_k^{-}$*, we cannot prove existence of bifurcation because there is a double eigenvalue of* $T(\nu)$ *crossing* 1*. In this case, a Hamiltonian-Hopf bifurcation may appear at* $\phi_k(a) = 1$*. This is described in Theorem 11.5.1 of [13], where two isolated solutions persist for* $\phi_k(a) > 1$*.*

3.2 Stability Analysis

Let

$$\sigma_m = \begin{cases} sgn(V''(a^2)) \text{ if } m \in [1, n/4) \\ -sgn(V''(a^2)) \text{ if } m \in (n/4, n/2] \end{cases}. \tag{22}$$

Theorem 7 *If* $\sigma_m < 0$*, or* $\sigma_m > 0$ *and* $\phi_1(a) < 1$*, then the relative equilibrium (2) is linearly stable.*

Proof. The Hamiltonian equation (8) is linearly stable at the equilibrium $\mathbf{a}_m$ if $\mathcal{J}D^2H(\mathbf{a}_m)$ has only pairs of purely conjugated imaginary eigenvalues $\pm i\nu$. Since $\mathcal{J}D^2H(\mathbf{a}_m)$ has a pair of zero eigenvalues that corresponds to the gauge symmetry, the system is linearly stable if $\mathcal{J}D^2H(\mathbf{a}_m)$ has $n-1$ pairs of purely imaginary eigenvalues.

The sign of ϕ_k does not depend on $k \in \{1, ..., n-1\}$ and is equal to σ_m. For σ_m negative, we have $\phi_k(a) < 1$. For σ_m positive, using the fact that $\cos k\zeta$ is increasing for $k \in [1, n/2] \cap \mathbb{N}$, then ϕ_k is decreasing for $k = 1, ..., n/2$ and $\phi_k(a) < \phi_1(a) < 1$. In both cases, it is not difficult to see that $\mathcal{J}D^2H(\mathbf{a}_m)$

has eigenvalues $i\nu_k^{\pm}$ for $k = 1, ..., n-1$. Since $\nu_{n-k}^{\pm} = -\nu_k^{\pm}$, we conclude that $\mathcal{J} D^2 H(\mathbf{a}_m)$ has $n-1$ pairs of purely imaginary eigenvalues. ■

4 Applications

In complex coordinates,

$$q_j(t) = e^{i\omega t} e^{ijm\zeta} \left(a + x_n(\nu t + jk\zeta)\right).$$

These solutions are discrete traveling waves in the sense that the norms satisfy

$$\left|q_j\right|(t) = a + \left|x_n\right|(\nu t + jk\zeta).$$

For example, if k divides n, the traveling wave has k identical waves with wavelength equal to n/k sites.

In the following sections, we present applications to the Schrödinger and Saturable lattice.

4.1 Schrödinger lattice

The cubic Schrödinger potential is $V(x) = cx^2/2$, where $c > 0$ corresponds to the focusing case, and $c < 0$ to the defocusing case. Standing waves given by (2) exist for

$$\omega = 4\sin^2 m\zeta/2 - ca^2.$$

In the focusing case, the potential satisfies

$$V''(a^2) = c > 0 \text{ and } \sigma = 1.$$

If $m \in [0, n/4)$, then $\alpha_k > 0$ and $\phi_k(a) < 1$ for $a^2 < \alpha_k/2c$. If $m \in (n/4, n/2]$, then $\alpha_k < 0$ and $\phi_k(a) < 1$ for all a.

Therefore, the following result holds.

Proposition 8 *In the focusing Schrödinger lattice ($c > 0$), the equilibrium $\mathbf{a}_m$ is linearly stable when $m \in (n/4, n/2]$, or $m \in [0, n/4)$ and $a < \sqrt{\alpha_1/2c}$. Moreover, the equilibrium $\mathbf{a}_m$ has two global bifurcations of traveling waves for each $k \in [1, n/2] \cap \mathbb{N}$ if $m \in (n/4, n/2]$, or $m \in [0, n/4)$ and $a < \sqrt{\alpha_k/2c}$.*

In the defocusing case, the potential satisfies

$$V''(a^2) = c < 0 \text{ and } \sigma = -1.$$

For $m \in [0, n/4)$, $\phi_k(a) < 1$. For $m \in (n/4, n/2]$, if $a^2 < \alpha_k/2c$, then $\phi_k(a) < 1$.

Proposition 9 *In the defocusing Schrödinger lattice ($c < 0$), the equilibrium $\mathbf{a}_m$ is linearly stable when $m \in [0, n/4)$, or $m \in (n/4, n/2]$ and $a < \sqrt{\alpha_1/2c}$. Moreover, the equilibrium $\mathbf{a}_m$ has two global bifurcations of traveling waves for each $k \in [1, n/2] \cap \mathbb{N}$ if $m \in [0, n/4)$, or $m \in (n/4, n/2]$ and $a < \sqrt{\alpha_k/2c}$.*

4.2 Saturable lattice

The Saturable potential is given by $V(x) = c \ln(1 + x)$ with $c > 0$. It is clear that

$$V''(a^2) = -c(1 + a^2)^{-2} \text{ and } \sigma = -1.$$

If $m \in [0, n/4)$, then $\phi_k(a) < 1$ for all a, while if $m \in (n/4, n/2]$, then $\phi_k(a) < 1$ for $(a + a^{-1})^{-2} < -\alpha_k/2c$.

Proposition 10 *The equilibrium $\mathbf{a}_m$ in the Saturable lattice is linearly stable when $m \in [0, n/4)$, or $m \in (n/4, n/2]$ and $(a + a^{-1})^{-1} < \sqrt{-\alpha_1/2c}$. Moreover, the equilibrium $\mathbf{a}_m$ has two global bifurcations of traveling waves for each $k \in [1, n/2] \cap \mathbb{N}$ if $m \in [0, n/4)$, or $m \in (n/4, n/2]$ and*

$$\left(a + a^{-1}\right)^{-1} < \sqrt{-\alpha_k/2c}.$$

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